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APPROXIMATE RAYLEIGH FORMULAS FOR THE FUNDAMENTAL FREQUENCY OF BAR-POINT MASS SYSTEMS

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1. INTRODUCTION

In a recent note, Yamamoto [1] has considered the well-known approximate Rayleigh-based formula

$$\omega = \bar{\mu}_R \sqrt{\frac{E}{\rho l^2}}, \quad \bar{\mu}_R = \sqrt{\frac{1}{1/3 + \gamma}} \tag{1}$$

due to Timoshenko [2], giving the fundamental frequency of the longitudinal vibrations of uniform bars fixed at one end and carrying a tip mass on the other, (where E is the Young's modulus of elasticity, ρ the mass density, l the length, A the cross-sectional area of the bar and $\gamma = M/\rho Al$ is the ratio of the tip mass to the bar mass) and has shown that this formula loses its accuracy for small values of γ.

This problem is actually a counterpart of the problem encountered in the Rayleigh-based formulas for the fundamental bending frequencies of beams carrying a point mass and is known to be ultimately related to the ill performance of the underlying mode shape estimate, as discussed by Low [3] and Turhan [4]. The problem is that, the strategy of approximating the fundamental mode shape by statically deformed shape due to the effect of the point mass weight, which is well suited in the cases where the initial effect on the point mass is dominant, loses its suitability with increasing inertial contribution of the elastic body itself.

A better strategy was shown [3, 4] to be that of using the statically deformed shape due to the combined effect of the elasticum and point mass weights. The purpose of the present note is to present alternative formulas based on that alternative strategy, for the fundamental longitudinal frequencies of bars with two different boundary conditions, carrying an arbitrarily located point mass. The performance of the presented formulas is tested by comparing them with the numerical solutions of the corresponding exact frequency equations.

2. ANALYSIS

It can be shown that the natural frequencies of the longitudinal vibrations of a bar carrying a point mass (Figure 1) can be calculated as

$$\omega = \mu \sqrt{\frac{E}{\rho l^2}}, \tag{2}$$

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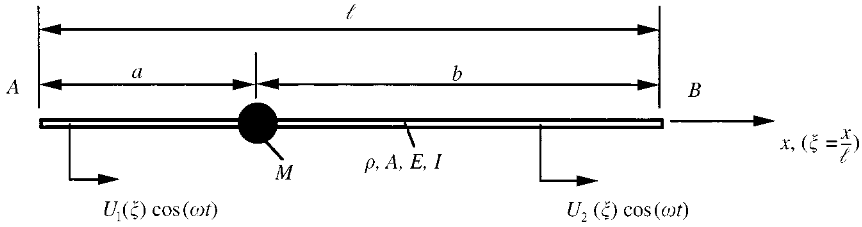

Figure 1. Bar-point mass system.

where μ is a root of the frequency equation

$$\cos(\mu) - \gamma\mu\sin(\alpha\mu)\cos(\mu\beta) = 0 \tag{3}$$

for a bar fixed at A and free at B [5], or

$$\sin(\mu) - \gamma\mu\sin(\alpha\mu)\sin(\mu\beta) = 0 \tag{4}$$

for a bar fixed at both A and B [6], where $\alpha = a/l$ is the non-dimensional location of the point mass and $\beta = 1 - \alpha$.

On the other hand, an approximation of μ can be calculated via Rayleigh method as

$$\mu_R = \sqrt{\frac{\int_0^\alpha U_1'^2(\xi)\mathrm{d}\xi + \int_\alpha^1 U_2'^2(\xi)\mathrm{d}\xi}{\int_0^\alpha U_1^2(\xi)\mathrm{d}\xi + \int_\alpha^1 U_2^2(\xi)\mathrm{d}\xi + \gamma U_{1,2}^2(\alpha)}}, \tag{5}$$

where $\xi = x/l$, $U_1(\xi)$ and $U_2(\xi)$ represent the estimated shape functions related to the considered mode at the left and right of the point mass, respectively, and primes denote derivatives with respect to ξ.

Adopting the above-mentioned strategy, i.e., using the statically deformed shape of the bar under the combined effect of its own weight and that of the point mass (acting both in the longitudinal direction) one has, for fixed–free bars

$$U_1(\xi) = \left(1 - \frac{\xi}{2}\right)\xi + \gamma\xi, \quad U_2(\xi) = \left(1 - \frac{\xi}{2}\right)\xi + \gamma\alpha \tag{6}$$

and

$$\mu_R = \sqrt{5\frac{1 + 3\gamma\alpha(2 - \alpha + \gamma)}{2 + 5\gamma\alpha[2 + 3(\gamma^2 + 3\gamma + 1)\alpha - (5\gamma + 4)\alpha^2 + \alpha^3]}}. \tag{7}$$

Similar calculations give

$$\bar{U}_1(\xi) = (1 - \xi)\xi + 2\gamma\beta\xi, \quad \bar{U}_2(\xi) = (1 - \xi)\xi + 2\gamma\alpha(1 - \xi) \tag{8}$$

TABLE 1

Fixed–free bar (Entries: (1) μ, (2) μ_R (3) % Error of μ_R)

$\gamma\backslash\alpha$	$\alpha = 0{\cdot}1$	$\alpha = 0{\cdot}2$	$\alpha = 0{\cdot}3$	$\alpha = 0{\cdot}4$	$\alpha = 0{\cdot}5$	$\alpha = 0{\cdot}6$	$\alpha = 0{\cdot}7$	$\alpha = 0{\cdot}8$	$\alpha = 0{\cdot}9$	$\alpha = 1$
	1) 1·570758	1·570646	1·570473	1·570254	1·570011	1·569769	1·569551	1·569377	1·569266	1·569227
0·001	2) 1·581143	1·581025	1·580826	1·580582	1·580324	1·580077	1·579863	1·579699	1·579595	1·579559
	3) 0·636	0·636	0·700	0·700	0·636	0·636	0·636	0·701	0·701	0·701
	1·570411	1·569294	1·567557	1·56538	1·562982	1·560597	1·558455	1·556754	1·55565	1·555245
0·01	1·581183	1·579998	1·578009	1·575583	1·57303	1·570603	1·568504	1·568887	1·565859	1·565485
	0·700	0·637	0·6317	0·702	0·639	0·640	0·706	0·642	0·642	0·643
	1·566886	1·555529	1·538313	1·517676	1·496129	1·475736	1·457977	1·443839	1·433979	1·42887
0·1	1·582091	1·569419	1·549412	1·526579	1·503909	1·483282	1·465838	1·452258	1·442956	1·438203
	0·957	0·835	0·715	0·592	0·534	0·474	0·548	0·554	0·627	0·629
	1·549859	1·489807	1·410567	1·332557	1·264592	1·20821	1·162492	1·126081	1·097811	1·076874
0·5	1·589857	1·511424	1·419538	1·336729	1·267449	1·211036	1·165765	1·129992	1·102434	1·082207
	2·580	1·409	0·637	0·300	0·158	0·248	0·344	0·355	0·364	0·464
	1·525167	1·401486	1·268854	1·160689	1·076874	1·011664	0·960259	0·919301	0·886506	0·860334
1	1·588482	1·419734	1·273449	1·162324	1·077998	1·012876	0·961739	0·921122	0·888711	0·862949
	4·131	1·356	0·315	0·086	0·092	0·098	0·208	0·217	0·225	0·348
	1·463476	1·232353	1·060184	0·943538	0·860334	0·797992	0·749527	0·710819	0·679296	0·653271
2	1·527114	1·239456	1·061338	0·943933	0·860663	0·798387	0·750022	0·711434	0·680043	0·654162
	4·374	0·568	0·094	0	0·116	0	0	0	0·147	0·153
	0·943328	0·675584	0·554591	0·482221	0·432841	0·396448	0·368232	0·345554	0·326832	0·311053
10	0·944257	0·675621	0·554599	0·482228	0·432851	0·396462	0·368249	0·345575	0·326857	0·311082
	0·106	0	0	0	0	0	0	0	0	0
	0·314724	0·222635	0·181845	0·157536	0·140952	0·128714	0·119205	0·111543	0·105199	0·099834
100	0·314724	0·222635	0·181845	0·157536	0·140952	0·128714	0·119205	0·111543	0·105199	0·099834
	0	0	0	0	0	0	0	0	0	0
	0·099953	0·07068	0·057712	0·049982	0·044706	0·040813	0·037786	0·035347	0·033327	0·031618
1000	0·099953	0·07068	0·057712	0·049982	0·044706	0·040813	0·037786	0·035347	0·033327	0·031618
	0	0	0	0	0	0	0	0	0	0

and

$$\mu_R = \sqrt{10\,\frac{1 + 12\gamma\delta(1+\gamma)}{1 + 10\gamma\delta[1 + 4\delta(1 + (3\gamma+4)\gamma)]}}, \tag{9}$$

where $\delta = \alpha\beta$ for fixed–fixed bars.

The results of equation (7) are compared with those of equation (3) for different values of γ and α in Table 1, and the results of equation (9) are compared with those of equation (4) in Table 2. An inspection of these tables shows that the accuracy of equations (7) and (9) are satisfactory for all practical purposes throughout the considered range of the parameters γ and α.

Returning now to the question raised by Yamamoto, substitute $\alpha = 1$ in equation (7) to obtain

$$\mu_R = \sqrt{5\,\frac{3\gamma^2 + 3\gamma + 1}{2 + 5\gamma[3\gamma^2 + 4\gamma + 2]}} \tag{10}$$

TABLE 2

Fixed–fixed bar (*Entries as Table* 1)

$\gamma\backslash\alpha$		$\alpha = 0{\cdot}1$	$\alpha = 0{\cdot}2$	$\alpha = 0{\cdot}3$	$\alpha = 0{\cdot}4$	$\alpha = 0{\cdot}5$
	1)	3·141292	3·140507	3·139537	3·138753	3·138454
0·001	2)	3·16205	3·161164	3·160153	3·159396	3·159119
	3)	0·668	0·636	0·636	0·637	0·669
		3·138573	3·13068	3·121045	3·113383	3·11049
0·01		3·159993	3·151076	3·141062	3·133662	3·13097
		0·669	0·638	0·640	0·674	0·675
		3·109592	3·027786	2·939675	2·878843	2·85774
0·1		3·137788	3·04484	2·955414	2·896564	2·876407
		0·900	0·561	0·510	0·625	0·629
		2·936987	2·552472	2·311557	2·190564	2·153748
0·5		2·965988	2·561082	2·322565	2·201505	2·164413
		0·987	0·352	0·475	0·502	0·464
		2·644381	2·109853	1·866243	1·753938	1·720667
1		2·652306	2·117735	1·87328	1·759651	1·725898
		0·302	0·379	0·375	0·342	0·290
		2·13055	1·624054	1·423005	1·332974	1·306542
2		2·135123	1·628525	1·425866	1·335003	1·308324
		0·187	0·307	0·210	0·150	0·076
		1·036	0·777441	0·678717	0·634923	0·622106
10		1·036626	0·777662	0·678827	0·634993	0·622165
		0·096	0·128	0	0	0
		0·332776	0·249583	0·217855	0·203785	0·199667
100		0·332779	0·249584	0·217855	0·203785	0·199667
		0	0	0	0	0
		0·105392	0·079044	0·068995	0·064539	0·063235
1000		0·105392	0·079044	0·068995	0·064539	0·063235
		0	0	0	0	0

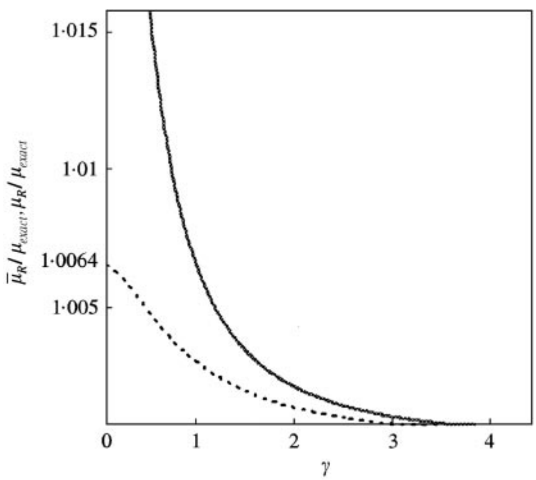

Figure 2. Comparison of equations (1) and (10). ($\bar{\mu}_R/\mu_{exact}$: ——; μ_R/μ_{exact}: - - - - - -).

as a counterpart of equation (1). The performance of equations (1) and (10) are compared in Figure 2 where $\bar{\mu}_R/\mu_{exact}$ and μ_R/μ_{exact} values are plotted versus γ. An inspection of this figure shows that the performance of equation (10) is remarkably superior to that of equation (1) and that this equation can reliably be used for small values of γ as well, with a maximum of 0·65% of error occurring at $\gamma = 0$.

3. FINAL REMARKS

Equations (7) and (9), in conjunction with equation (2) can reliably be used to predict the fundamental frequencies of the longitudinal vibrations of bars carrying an arbitrarily located point mass.

Due to the mathematical equivalence of the two problems, the same equations can be used to predict the fundamental frequencies of the torsional vibrations of bars carrying an arbitrarily located disc as well. To this end it suffices to consider

$$\omega = \mu \sqrt{\frac{G}{\rho l^2}} \tag{11}$$

instead of equation (2) and take γ as the ratio of the mass moment of inertia of the attached disc to that of the bar.

Finally, due to the analogy between (fixed–free bar)-(tip mass) and spring–mass systems, μ_R of equation (10) can be used instead of $\bar{\mu}_R$ of equation (1) to predict the frequency of spring–mass systems with the mass of the spring approximately taken into consideration. In this case, the frequency formula should be of course written as

$$\omega = \mu_R \sqrt{\frac{k_s}{m_s}}, \tag{12}$$

where k_s is the stiffness coefficient and m_s is the mass of the spring.

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